Dynamical Symmetry Group Based on Dirac Equation and Its Generalization to Elementary Particles*

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(Received 30 January 1964)

The dynamical or exact symmetry group of the nonrelativistic Kepler problem (a symmetry group in four dimensions) is generalized to the Dirac equation and further to elementary particles. The former is a ten-parameter group of rank two isomorphic to a group in five dimensions, the latter a 16-parameter group of rank four isomorphic to a group in six dimensions. Both groups contain the real Lorentz group and couple the space-time quantum numbers with the internal quantum numbers. The 16-parameter group has a 15-parameter simple subgroup and contains two three-dimensional rotation groups, one for ordinary spin and one for isotopic spin. The concept of "inhomogeneous dynamical group" is introduced. The inhomogeneous group contains two new additive quantum numbers to describe the hypercharge and the baryon number and leads to a mass spectrum. The third component of isospin and the new additive quantum numbers commute with all the six generators of the Lorentz group. A further generalization leads to a group where all three isospin generators commute with the Lorentz group.

I. INTRODUCTION

BY an exact or dynamical symmetry group of a quantum-mechanical system we mean a group which, above and beyond the space-time symmetry, gives the actual quantum numbers and degeneracy of the system. It has been called in the literature variously as the "hidden" or "accidental" symmetry. It has been known for a long time that the dynamical group of the nonrelativistic Kepler problem is a six parameter group of rank two whose generators can be taken to satisfy the commutation relations of the four dimensional orthogonal group[1–6] or those of the Lorentz group.[7] The symmetric irreducible representations of this larger group explain the n^2 fold degeneracy of the nth level of the hydrogen atom.[8] Similarly the dynamical symmetry group of an N-dimensional isotropic harmonic oscillator is the group U_N, the N-dimensional unitary group,[9] whose irreducible representations of dimension $(N+n-1)!/n!(N-1)!$ explain the degeneracy of the nth energy level.

In this paper we discuss the generalizations of these dynamical groups applicable to the relativistic case, the Dirac electron, and further to elementary particles. We shall give to the concept of dynamical symmetry a more fundamental meaning than a mere accident. The underlying larger space may be thought to contain beside the space time coordinates the internal coordinates (or alternately as some topological deviations from the flat space) which give rise to the observed quantum numbers, just as the space-time coordinates give rise to spin. In the simple cases mentioned above, the H-atom or oscillator, the dynamical groups can be explicitly constructed in terms of the position and momentum operators. However, a quantum-mechanical problem may be completely defined by its dynamical symmetry group, the class of representations which are realized and by the connection between the energy (or mass) and the invariant (Casimir) operator of the dynamical group. In this sense, in generalizing the dynamical symmetry groups to elementary particles we shall be guided by the group structure and by the quantum numbers one obtains; specifically the larger group must contain the Lorentz group and, coupled to it, all other observed quantum numbers.[10]

One generalization of the nonrelativistic dynamical symmetry groups is the 16-parameter complex Lorentz group with a real metric and has been discussed in great detail elsewhere.[11] It contains the real Lorentz group, the symmetry group of the three-dimensional harmonic oscillator, U_3, and as a limiting case, the symmetry group of the Kepler problem.

A second natural generalization is the subject of this paper. We determine first a ten-parameter group generated by the four Dirac matrices γ_μ which is isomorphic to the orthogonal group in five dimensions with the metric $(-,+,-,-,-)$. It contains the seven-parameter subgroup which is actually the dynamical symmetry group of the nonrelativistic Kepler problem. We then discuss the symmetry group generated by the 16 Dirac matrices; it is a group isomorphic to the orthogonal group in 6 dimensions with the metric $(-,-,+,-,-,-)$. It contains two rotation groups, one for spin and one for isospin, and the inhomogeneous group contains beside the energy momentum operators two other

* Supported in part by the U. S. Air Force Office of Scientific Research and the National Science Foundation.

[1] W. Pauli, Z. Physik **36**, 336 (1926).
[2] V. Fock, Z. Physik **98**, 145 (1935).
[3] V. Bargmann, Z. Physik **99**, 576 (1936).
[4] O. Laporte and G. Y. Rainich, Trans. Am. Math. Soc. **39**, 154 (1936).
[5] S. P. Alliluev, Zh. Eksperim. i Teor. Fiz. **33**, 200 (1957) [English transl.: Soviet Phys.—JETP **6**, 156 (1958)].
[6] H. V. McIntosh, Am. J. Phys. **27**, 620 (1959).
[7] This observation is due to O. Klein, see L. Hulthen, Z. Physik **86**, 21 (1933).
[8] The irreducible representations of this group are characterized by two integer or half-integer numbers n and m; the dimensionality is $(2n+1)(2m+1)$. Only the representations $n=m$ are realized in the Kepler problem.
[9] J. M. Jauch and E. L. Hill, Phys. Rev. **57**, 641 (1940); G. A. Baker, Jr., Phys. Rev. **103**, 1119 (1956); and Ref. 5.
[10] For a discussion of the coupling of space-time quantum numbers and the internal quantum numbers see A. O. Barut, Nuovo Cimento **32**, 234 (1964).
[11] A. O. Barut (to be published).

additive quantum numbers to describe the hypercharge Y and the baryon number N. The third component of isospin, as well as the two other additive quantum numbers commute with all the six generators of the real Lorentz group. The implications for the mass spectrum of elementary particles are discussed.

A further generalization to the orthogonal group in 7 dimensions with the metric $(- - - + - - -)$ leads to the case where all three generators of isospin commute with all the generators of the Lorentz group. It should be remarked that a dynamical group should give an exact mass spectrum and is therefore different than symmetry considerations based on compact groups, such as SU_3, independent of the Lorentz group, which can only give a very approximate or broken symmetry.

II. THE LIE GROUP GENERATED BY THE DIRAC MATRICES

Let us consider the four Dirac matrices as generators of a Lie group. What is the group? The commutation relations of these matrices lead to new generators. For example, in the representation in which

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}, \tag{2.1}$$

we find

$$[\alpha_i,\alpha_j]=2iA_k,\ (ijk\text{ cyclic});\quad [\alpha_i,\beta]=2B_i, \tag{2.2}$$

where

$$\mathbf{A}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}0 & -\boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}. \tag{2.3}$$

These ten independent generators $\boldsymbol{\alpha}$, β, $\mathbf{A}$, $\mathbf{B}$ now form a Lie algebra because their commutation relations do not give anything new:

$$[A_i,B_j]=2iB_k;\quad [A_i,A_j]=2iA_k;\quad [B_i,B_j]=-2iA_k\quad (ijk\text{ cyclic}). \tag{2.4}$$

Moreover, as seen from (2.4) the 6 generators $\mathbf{A}$ and $\mathbf{B}$ form a subalgebra or generate a subgroup. Let us introduce the new generators

$$\mathbf{R}=\frac{1}{2i}\mathbf{A};\quad \mathbf{M}=\frac{1}{2i}\mathbf{B};\quad \mathbf{L}=-\tfrac{1}{2}\boldsymbol{\alpha},\quad N_{00}=-i\beta. \tag{2.5}$$

Then $\mathbf{R}$ and $\mathbf{L}$, and also $\mathbf{R}$ and $\mathbf{M}$, satisfy the commutation relations of the Lorentz group with $\mathbf{R}$ being the generators of the subgroup of three-dimensional rotations.

We now show the relation of this group to the group of complex Lorentz transformations Λ satisfying $\Lambda^\dagger G\Lambda=G$, where G is the diagonal matrix with elements $(+1,-1,-1,-1)$. This latter group is generated by the following 16 generators[11]:

$$M_{\mu\nu}=\begin{bmatrix}0 & L_2 & L_3 & L_1\\ & 0 & R_1 & R_3\\ & & 0 & R_2\\ & & & 0\end{bmatrix}=-M_{\nu\mu};$$

$$N_{\mu\nu}=\begin{bmatrix}N_{00} & M_2 & M_3 & M_1\\ & N_{11} & U_1 & U_3\\ & & N_{22} & U_2\\ & & & N_{33}\end{bmatrix}=N_{\nu\mu}, \tag{2.6}$$

with the commutation relations

$$\begin{aligned}
[M_{\mu\sigma},M_{\mu\nu}]&=-g_{\nu\sigma}M_{\mu\rho}-g_{\mu\rho}M_{\nu\sigma}+g_{\mu\sigma}M_{\nu\rho}+g_{\nu\rho}M_{\mu\sigma},\\
[N_{\mu\nu},N_{\sigma\rho}]&=g_{\nu\sigma}M_{\mu\rho}+g_{\mu\rho}M_{\nu\sigma}+g_{\mu\sigma}M_{\nu\rho}+g_{\nu\rho}M_{\mu\sigma},\\
[M_{\mu\nu},N_{\sigma\rho}]&=-g_{\nu\sigma}N_{\mu\rho}+g_{\mu\rho}N_{\nu\sigma}+g_{\mu\sigma}N_{\nu\rho}-g_{\nu\rho}N_{\mu\sigma}.
\end{aligned} \tag{2.7}$$

If we set in these commutation relations the generators $\mathbf{U}$ and N_{ii} $(i=1,2,3)$ identically equal to zero, we obtain again the commutation relations of the Lie Algebra of the Dirac matrices. In other words, the underlying complex space is such that only the x^0 component has an imaginary part: x^0+iy^0.

It will be shown now that, alternatively, the Lie group of the Dirac matrices is isomorphic to the five-dimensional real "Lorentz" group with the metric $(-1,+1,-1,-1,-1)$. For this purpose we introduce the following antisymmetric set of generators

$$M_{ab}=\begin{bmatrix}0 & iN_{00} & iM_2 & iM_3 & iM_1\\ & 0 & L_2 & L_3 & L_1\\ & & 0 & R_1 & R_3\\ & & & 0 & R_2\\ & & & & 0\end{bmatrix}
=\frac{1}{2}\begin{bmatrix}0 & -\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3\\ & 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3\\ & & 0 & \gamma_2\gamma_1 & \gamma_3\gamma_1\\ & & & 0 & \gamma_3\gamma_2\\ & & & & 0\end{bmatrix}. \tag{2.8}$$

Then the commutation relations become

$$[M_{ab},M_{cd}]=-g_{bc}M_{ad}-g_{ad}M_{bc}+g_{ac}M_{bd}+g_{bd}M_{ac}$$
$$a,b=1,2,3,4,5;\quad g_{ab}=(-1,+1,-1,-1,-1), \tag{2.9}$$

which are the commutation relations of the real "Lorentz" group with the metric g_{ab}. We shall discuss the structure constants and the invariant operator of this group in connection with the larger group in the next section.

A third convenient way of representing the generators is by means of an antisymmetric tensor $M_{\mu\nu}$ and a four-vector l_μ

$$M_{\mu\nu}=\tfrac{1}{2}(\gamma_\mu\gamma_\nu-g_{\mu\nu}),\quad l_\mu=\gamma_\mu/2 \tag{2.10}$$

with the commutation relations

$$\begin{aligned}
[l_\mu,l_\nu]&=M_{\mu\nu},\\
[M_{\mu\nu},l_\sigma]&=g_{\nu\sigma}l_\mu-g_{\sigma\mu}l_\nu,\\
[M_{\mu\nu},M_{\sigma\rho}]&=-g_{\nu\sigma}M_{\mu\rho}-g_{\mu\rho}M_{\nu\sigma}+g_{\mu\sigma}M_{\nu\rho}+g_{\nu\rho}M_{\mu\sigma}.
\end{aligned} \tag{2.11}$$

III. THE LIE GROUP GENERATED BY DIRAC-CLIFFORD MATRICES

We now consider the 15 Dirac matrices as generators of a Lie group. If we order these matrices in the form

$$M_{ab}=\tfrac{1}{2}\begin{bmatrix} 0 & \gamma_5 & -\gamma_5\gamma_0 & \gamma_5\gamma_1 & \gamma_5\gamma_2 & \gamma_5\gamma_3 \\ & 0 & -\gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ & & 0 & \gamma_0\gamma_1 & \gamma_0\gamma_2 & \gamma_0\gamma_3 \\ & & & 0 & \gamma_2\gamma_1 & \gamma_3\gamma_1 \\ & & & & 0 & \gamma_3\gamma_2 \\ & & & & & 0 \end{bmatrix} \qquad (3.1)$$

we have again the commutation relations

$$[M_{ab},M_{cd}]=-g_{bc}M_{ad}-g_{ad}M_{bc}+g_{ac}M_{bd}+g_{bd}M_{ac}\,,$$
$$a,\, b=1,\, 2,\, \cdots\, 16;$$
$$g_{ab}=(-1,\,-1,\,+1,\,-1,\,-1,\,-1)\,, \qquad (3.2)$$

which are those of the Lorentz group in six dimensions with the metric $(-1,\,-1,\,+1,\,-1,\,-1,\,-1)$. The structure constants are given by

$$(C_{ab})_{cd}{}^{ef}=-g_{bc}(g_a{}^e g_d{}^f-g_a{}^f g_d{}^e)-g_{ad}(g_b{}^e g_c{}^f-g_b{}^f g_c{}^e)$$
$$+g_{ac}(g_b{}^e g_d{}^f-g_b{}^f g_d{}^e)+g_{bd}(g_a{}^e g_c{}^f-g_a{}^f g_c{}^e)\,. \qquad (3.3)$$

The invariant or the Casimir operator of the homogeneous group can be derived from the structure constants by first constructing a "metric" tensor in the space of the generators defined by[12]

$$g_{ab,\alpha\beta}=(C_{ab})_{cd}{}^{ef}(C_{\alpha\beta})_{ef}{}^{cd}\,, \qquad (3.4)$$

with the result that

$$g_{ab,\alpha\beta}=\mathrm{const}(g_{a\beta}g_{b\alpha}-g_{a\alpha}g_{b\beta})\,. \qquad (3.5)$$

We see that

$$\det(g_{ab,\alpha\beta})\neq 0\,, \qquad (3.6)$$

so that the group is at least semisimple. The Casimir operator is then given by

$$F^2=(g^{a\beta}g^{b\alpha}-g^{a\alpha}g^{b\beta})M_{ab}M_{\alpha\beta}$$
$$=2\,\mathrm{tr}(MGMG)\,, \qquad (3.7)$$

or, in terms of the generators (2.8) by

$$F^2=2(\mathbf{M}^2+\mathbf{L}^2-\mathbf{R}^2-N_{00}{}^2$$
$$-M_{12}{}^2+M_{13}{}^2-M_{14}{}^2-M_{15}{}^2-M_{16}{}^2)\,. \qquad (3.8)$$

In the case of the 10-parameter group it takes the form

$$F^2=2(\mathbf{M}^2+\mathbf{L}^2-\mathbf{R}^2-N_{00}{}^2)\,, \qquad (3.9)$$

which is indeed the limiting case of the invariant operator of the complex Lorentz group.[11]

A second invariant operator is given by

$$G^4=\mathrm{tr}(MGMGMGMG) \qquad (3.10)$$

which reduces, in the case of the 10-parameter group to

$$G^4=-(\mathbf{M}\cdot\mathbf{R})^2-(\mathbf{R}\cdot\mathbf{L})^2+[N_{00}\mathbf{R}-(\mathbf{M}\times\mathbf{L})]^2\,. \qquad (3.11)$$

[12] G. Racah, Institute for Advanced Study Lecture Notes (1951) (Reprinted CERN 61-3); W. Pauli, Lecture Notes, CERN-31; A. Salam in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963).

Again a convenient way of representing the generators is in terms of one antisymmetric tensor $M_{\mu\nu}=\frac{1}{2}(\gamma_\mu\gamma_\nu-g_{\mu\nu})$ two four-vectors $l_\mu=\frac{1}{2}\gamma_\mu$, $l_\mu'=\frac{1}{2}\gamma_5\gamma_\mu$, and one scalar $K=\gamma_5/2$ with the commutation relations (2.11) between $M_{\mu\nu}$ and l_μ plus the following:

$$\begin{aligned}[M_{\mu\nu},l_\sigma']&=g_{\nu\sigma}l_\mu'-g_{\sigma\mu}l_\nu'\,,\\ [l_\mu',l_\nu']&=M_{\mu\nu}\,,\\ [l_\mu,l_\nu']&=-g_{\mu\nu}K\,,\\ [K,l_\mu]&=l_\mu'\,,\\ [K,l_\mu']&=-l_\mu\,,\\ [K,M_{\mu\nu}]&=0\,.\end{aligned} \qquad (3.12)$$

The Clifford algebra has actually 16 elements. But one generator commutes with all others and is equal to a multiple of identity in every irreducible representation. The remaining 15-parameter group is simple. This situation is exactly the same as in complex Lorentz group.[11]

Finally, we discuss the inhomogeneous group. There are six generators for the translations. Four corresponding to space-time coordinates must be identified with the energy momentum vector of the composite system, k_μ, and two others which we denote by h_1 and h_2. The commutation relations are with $k_a=(h_1,h_2,k_\mu)$,

$$\begin{aligned}[M_{ab},k_c]&=g_{ac}k_b-g_{bc}k_a\,,\\ [k_a,k_b]&=0\,.\end{aligned} \qquad (3.13)$$

The invariant of the inhomogeneous group is no longer k^2, but

$$k^2+h_1{}^2+h_2{}^2=C\,. \qquad (3.14)$$

The same commutation relations (3.2), (3.13) hold in the case of larger group in 7 dimensions.

IV. THE PHYSICAL INTERPRETATION OF THE GENERATORS

According to general principles of quantum theory, extended now to the larger dynamical symmetry group, the infinitesimal generators of the group will be identified with the quantum-number operators whose possible values will be determined by their eigenvalues in a given irreducible representation.

As we mentioned before the generators $M_{\mu\nu}$ in Eq. (2.10), or the generators $\mathbf{R}$ and $\mathbf{L}$ in the notation of Eq. (2.8) span the real Lorentz group. This is the space-time subgroup of the dynamical symmetry. Accordingly, we must identify the three-dimensional rotation group generated by $\mathbf{R}$ with the spin or angular momentum group. This particular subgroup is common to all problems, in particular, all special systems mentioned in the Introduction. The remaining generators correspond to quantum numbers of dynamical origin from the usual point of view. Thus, the generators $\mathbf{M}$ describe the Kepler problem together with $\mathbf{R}$ and N_{00}. In the relativistic case we add to these the remaining generators $\mathbf{L}$ of the Lorentz group. The resultant 10-parameter group, discussed in Sec. II, is of rank two. That is, there are two mutually commuting generators, which can be taken as the third component of angular momentum R_3 and N_{00}. The energy levels will then be given in terms of two quantum numbers as is well known. We note that in the nonrelativistic case (**L** put identically equal to zero), N_{00} commutes with all the six generators **R** and **M** and therefore it is equal to a multiple of identity in any irreducible representation. The remaining group is simple and of rank two.

The full 16-parameter group of Sec. III is of rank four. One of the generators is however again equal to the identity so that the 15-parameter simple group has three mutually commuting generators or quantum numbers. Furthermore, we notice that the generators M_{12}, $-iM_{13}$, and $-iM_{23}$ span another three-dimensional rotation group disjoint from the spin group. We can identify it with the group of isotopic spin **I**.

Although both the complex Lorentz group and the group presented here are 16-parameter groups of rank four with a 15-parameter simple subgroups with a lot of generators in common, there are some differences. One generalizes the Kepler problem, the other the oscillator type of problems. In the case of the 7-dimensional group the isotopic spin generators all commute with the Lorentz group.

The identification of the dynamical symmetry group is only part of the problem. The more important task is the determination of energy or mass levels in terms of the quantum numbers determined by the group. This question is discussed in the next section.

V. THE ENERGY AND MASS SPECTRUM

In nonrelativistic problems where one can explicitly construct the generators in terms of the variables entering the Hamiltonian the energy, being a number in a given irreducible representation, is a function of the Casimir operator of the homogeneous dynamical group:

$$E=f(F^2). \tag{5.1}$$

Thus, for a rotator one has trivially

$$E=\mathbf{J}^2, \tag{5.2}$$

in proper units; for the two-dimensional isotropic harmonic oscillator

$$E=2\lambda(\mathbf{J}^2+1/4)^{1/2}, \quad \lambda=\hbar\omega, \tag{5.3}$$

(the dynamical group in this case is U_2, the four generators of U_2 can be grouped into the three angular momentum operators J and one other generator which is a multiple of identity). For the Kepler problem one obtains[1–6]

$$E=\lambda^2(\mathbf{M}^2-\mathbf{R}^2-I)^{-1}, \quad \lambda^2=Z^2e^4m/\hbar^2. \tag{5.4}$$

The Casimir operator in this case reduces by (3.9) to $\mathbf{M}^2-\mathbf{R}^2-N_{00}^2$, but N_{00} can be chosen to be the identity. From the commutation relations of **M** and **R** one finds[7] that the eigenvalues of $\mathbf{M}^2-\mathbf{R}^2$ are $-(n^2-1)$ so that the eigenvalue of the invariant operator in (5.4) is $F^2=-n^2$. The occurrence of N_{00}, therefore of the seven-parameter dynamical group is thus essential in accounting for correct degeneracy and the correct value of the energy levels.

In the case of the Dirac electron the explicit form of the Eq. (5.1) is not known. We are checking this relation within the Hamiltonian formalism in terms of the invariant of the 10-parameter group discussed in Sec. II.

We write Eq. (5.1) for relativistic bound-state problems conveniently in the form

$$m^2=f(F^2), \tag{5.5}$$

where m is the rest mass of the system. We can understand the occurrence of this fundamental relation by requiring an invariance under the *inhomogeneous dynamical symmetry*. In the case of relativistic mass points the dynamics is really contained in the inhomogeneous Lorentz group. The subgroup of translations leads to the concept of energy momentum fourvector. Now we have an invariance with respect to transformations in a larger space. If we introduce translations in the space-time coordinates x^μ alone and not the remaining "internal" coordinates, the square of the energy momentum vector k^2 would be an invariant and no mass quantization can be obtained. It is thus natural to introduce the full inhomogeneous group in which case the invariant is

$$k^2+h^2=m^2+h^2=C, \tag{5.10}$$

where h^2 is the square of a four-vector for the complex Lorentz group,[11] of a two component quantity in the case of the group of Sec. III, Eq. (3.14). The two generators h_1 and h_2 commute, by Eq. (3.13), with all the six generators of the Lorentz group. They correspond to additive quantum numbers and can be identified with the hypercharge and baryon number. The inhomogeneous group has other invariants involving the generators of the homogeneous group. An irreducible representation of the inhomogeneous group is characterized by fixed values of these invariants; then it would be possible to determine the eigenvalues of h^2, and consequently, by Eq. (5.10), the quantized mass states of the composite system. This is the program proposed to determine the energy and mass spectrum of quantum mechanical systems once the full dynamical group is known. It would lead, if successful, to an alternate formulation of quantum theory with no Hamiltonian or space-time coordinates.[13]

[13] For phenomenological mass formulas resulting from (5.5) see A. O. Barut, in *Proceedings of the Conference on Symmetry Principles at High Energy* (W. H. Freeman and Company, New York, 1964).